Thermodynamics of f(R) Gravity with the Disformal Transformation

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- Thermodynamics in Jordan Frame
  - Non-equilibrium description
  - Equilibrium description
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What is f(R) theory?

- Einstein equation can be obtained by varying the Einstein-Hilbert action:

$$S_{EH}=\frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\,R+S_{\mathrm{M}}[\mathrm{g}_{\mu\nu},\Psi],$$

$$\delta S_{EH}=0\;\Rightarrow G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\kappa T_{\mu\nu}.$$

- General Relativity(GR) becomes f(R) theory by replacing the Lagrangian density of GR, R, with a function of R, $f(R)$:

$$S=\frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\,f(R)+S_{\mathrm{M}}[\mathrm{g}_{\mu\nu},\Psi],$$

$$\delta S=0\Rightarrow FG_{\mu\nu}-\frac{1}{2}g_{\mu\nu}(f(R)-FR)-\nabla_\mu\nabla_\nu F+g_{\mu\nu}\Box F=\kappa T_{\mu\nu}.\;(F\coloneqq f_{,R})$$

Jordan Frame v.s. Einstein Frame

- f(R) gravity in Jordan frame:

$$S = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\; f(R)$$

Conformal transformation: $\tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}, \quad \Omega^2(x) = F(R) := e^{\sqrt{\frac{4\pi G}{3}}\omega}$

- f(R) gravity in Einstein frame:

$$S = \int d^4x\,\sqrt{-\tilde{g}}\left[\frac{1}{2\kappa}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\omega\partial_\nu\omega - V(\omega)\right],$$

where $V(\omega) = \frac{1}{2\kappa}\frac{FR-f}{F^2}$.

Disformal Transformation [arXiv: gr-qc/9211017]

- In 1992, Bekenstein proposed a new gravity theory which is a special kind of bimetric theory.
- One of the metric, $g_{\mu\nu}$, describes the gravitational field of the spacetime, the other, $\gamma_{\mu\nu}$, describes the particle trajectory in gravitational field.
- Bekenstein argued that two metrics should be related through disformal transformation in order to satisfy equivalence principle and causality:

$$\gamma_{\mu\nu} = A(\phi, X) g_{\mu\nu} + B(\phi, X)\partial_\mu\phi\partial_\nu\phi. \qquad \left(X = -\frac{1}{2} g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right)$$

matter (physical) metric

gravitational metric

gravitational scalar field

Thermodynamics and Gravity

- In BH physics, the temperature and entropy are associated with the surface gravity and area of the horizon. [Comm. Math. Phys., Volume 31, Number 2 (1973), 161-170.]
- In 1995, T. Jacobson further showed that Einstein equation can be derived from the thermodynamic behavior of spacetime.[arXiv:gr-qc/9504004v2]
- In 2005, Cai and Kim demonstrated that the Friedmann equations can be derived from the first law of thermodynamics on the apparent horizon of the universe.[arXiv:gr-qc/0611071v2]
- Connection between thermodynamics and f(R) gravity and other modified gravity has been widely investigated. [arXiv:0909.2109, 1005.5234]

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Action: $S = \frac{1}{2\kappa}\int \sqrt{-g}\, f(R) d^4x \ + \sum_i S_{\mathrm{M}}^{(i)}[\gamma_{\alpha\beta}]$

- $\gamma_{\alpha\beta}$ is related with $g_{\alpha\beta}$ by the disformal transformation:

$$\gamma_{\alpha\beta} = A(\phi, X) g_{\alpha\beta} + B(\phi, X)\partial_\alpha\phi\partial_\beta\phi$$

where $X = -\frac{1}{2}\, g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi$ is the kinetic term of disformal field ϕ.

- Under the assumption of Principle of Cosmology :

➢ Metric $g_{\alpha\beta}$ is given by Robertson-Walker metric: (k=0)

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2)$$

➢ ϕ depends only on time now, i.e., $\phi = \phi(t)$.

Action: $S = \frac{1}{2\kappa}\int \sqrt{-g}\, f(R) d^4x \; + \sum_i S_{\mathrm{M}}^{(i)}[\gamma_{\alpha\beta}]$

- $\gamma_{\alpha\beta}$ is related with $g_{\alpha\beta}$ by the disformal transformation:

$$\eta_{\alpha\beta} = A(\phi, X) g_{\alpha\beta} + B(\phi, X)\partial_\alpha\phi\partial_\beta\phi$$

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Equation of Motion

- $F G_{\mu\nu} = \kappa \sum_i \left(T^{(i)}_{\mu\nu} + t^{(i)}_{\mu\nu} \right) + \kappa \hat{T}^{(\mathrm{d})}_{\mu\nu}$, $F \coloneqq \partial f(R)/\partial R$
- $\dfrac{\partial \mathcal{L}_i}{\partial \phi} + \dfrac{(\partial_\alpha \phi)(4\partial_\phi A - 2X\partial_\phi B)}{(2X)(-4A_{,X} + 2B + 2XB_{,X})} \dfrac{\partial \mathcal{L}_i}{\partial(\partial_\alpha \phi)} = 0$

where

$$T^{(i)}_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta \mathcal{L}_i}{\delta g^{\mu\nu}},$$

$$t^{(i)}_{\mu\nu} = \frac{-2}{\sqrt{-g}} \left(\frac{-A g_{\mu\nu} - 2A_{,X}\partial_\mu\phi\partial_\nu\phi + XB_{,X}\partial_\mu\phi\partial_\nu\phi}{(2X)(-4A_{,X} + 2B + 2XB_{,X})} \right) (\partial_\alpha \phi) \frac{\partial \mathcal{L}_i}{\partial(\partial_\alpha \phi)},$$

$$\hat{T}^{(\mathrm{d})}_{\mu\nu} = \frac{1}{\kappa} \left[\frac{1}{2} g_{\mu\nu}(f(R) - FR) + \nabla_\mu \nabla_\nu F - g_{\mu\nu} \Box F \right]$$

- With the perfect fluid assumption, one is able to express induced matter in terms of ordinary matter:

$$\rho_i^{(\mathrm{in})} = \lambda \rho_i,\ \ P_i^{(\mathrm{in})} = \lambda w^{(\mathrm{in})} \rho_i$$

where

$$\lambda = \frac{(1 - a^2)(3\dot{a}\dot{\phi} + a^3\ddot{\phi})}{3\dot{a}\dot{\phi}}$$

and

$$w^{(\mathrm{in})} := \frac{P_i^{(\mathrm{in})}}{\rho_i^{(\mathrm{in})}} = -\frac{a\ddot{\phi}}{3\dot{a}\dot{\phi} + a^3\ddot{\phi}}.$$

- Assume that ordinary matter only contain non-relativistic matter(m) and radiation(r).
  - $\rho_i = \rho_\mathrm{m} + \rho_\mathrm{r} := \bar{\rho}_\mathrm{M}$
  - $\rho_i^{(\mathrm{in})} = \rho_\mathrm{m}^{(\mathrm{in})} + \rho_\mathrm{r}^{(\mathrm{in})} = \lambda\,(\rho_\mathrm{m} + \rho_\mathrm{r}) := \lambda \bar{\rho}_\mathrm{M}$
  - $P_i = P_\mathrm{r}$
  - $P_i^{(\mathrm{in})} = P_\mathrm{m}^{(\mathrm{in})} + P_\mathrm{r}^{(\mathrm{in})} = \lambda w^{(\mathrm{in})}(\rho_\mathrm{m} + \rho_\mathrm{r}) := \lambda w^{(\mathrm{in})} \bar{\rho}_\mathrm{M}$

Modified Friedmann Equations (Non-equilibrium case)

- $H^2 = \frac{8\pi G}{3F}(\bar{\rho}_\mathrm{M} + \hat{\rho}_\mathrm{d} + \lambda\bar{\rho}_\mathrm{M}) \coloneqq \frac{8\pi G}{3F}\hat{\rho}_t,$
- $\dot{H} = -\frac{4\pi G}{F}[(\bar{\rho}_\mathrm{M} + \hat{\rho}_\mathrm{d} + P_\mathrm{r} + \hat{P}_\mathrm{d}) + \lambda\left(1 + w^{(\mathrm{in})}\right)\bar{\rho}_\mathrm{M}$
  $\coloneqq \frac{-4\pi G}{F}(\hat{\rho}_t + \hat{P}_t),$

where

$$\hat{\rho}_\mathrm{d} = \frac{1}{8\pi G}\left(\frac{1}{2}(FR - f) - 3H\dot{F}\right),$$

$$\hat{P}_d = \frac{1}{8\pi G}\left(\ddot{F} + 2H\dot{F} - \frac{1}{2}(FR - f)\right)$$

are the dark energy density and pressure, respectively.

1st & 2nd Law of Thermodynamics (Non-equilibrium case)

- $\dot{\hat{\rho}}_{\rm t} + 3H(\hat{\rho}_{\rm t} + \hat{P}_{\rm t}) = \frac{3H^2\dot{F}}{8\pi G}$ $\Longrightarrow$ Non-equilibrium Thermodynamics!

Temp. on the horizon

$\hat{S}$: Horizon entropy in f(R)

$\widehat{W}$: Work density in non-equil. picture

- First law: $Td\hat{S} + Td_i\hat{S} = -d\hat{E} + \widehat{W}dV,$

entropy production term in non-equil. thermodynamics

$\hat{E} = V\hat{\rho}_t$

- Second law: $\frac{d\hat{S}}{dt} + \frac{d_i\hat{S}}{dt} + \frac{d\hat{S}_t}{dt} = \frac{12\pi F\dot{H}^2}{GRH^3} \geq 0.$

Entropy change rate of ordinary matter and induced matter within the horizon

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Modified Friedmann Equation (Equilibrium case)

- $H^2 = \frac{8\pi G}{3}(\bar{\rho}_{\mathrm{M}} + \rho_{\mathrm{d}} + \lambda\bar{\rho}_{\mathrm{M}}) := \frac{8\pi G}{3}\rho_t$
- $\dot{H} = -4\pi G[(\bar{\rho}_{\mathrm{M}} + \rho_{\mathrm{d}} + P_r + P_{\mathrm{d}}) + \lambda(1 + w^{(\mathrm{in})})\bar{\rho}_{\mathrm{M}}]$

  $:= -4\pi G(\rho_t + P_t)$

where

$$\rho_{\mathrm{d}} = \frac{1}{8\pi G}\left(\frac{1}{2}(FR - f) - 3H\dot{F} + 3H^2(1 - F)\right),$$

$$P_d = \frac{1}{8\pi G}\left(\ddot{F} + 2H\dot{F} - \frac{1}{2}(FR - f) - (1 - F)(2\dot{H} + 3H^2)\right)$$

are the dark energy density and pressure in equilibrium picture, respectively.

1st & 2nd Law of Thermodynamics (Equilibrium case)

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- $\rho_t + 3H(\rho_t + P_t) = 0$ $\Longrightarrow$ Equilibrium Thermodynamics

Horizon entropy in equil. description

W: Work density in equil. picture

- First law: $TdS = -dE + WdV$

Temp. on the horizon

$E = \rho_t V$(total energy in equil. picture)

Entropy change rate of ordinary matter and induced matter within the horizon

- Second law : $\frac{d}{dt}(S + S_t) = \frac{12\pi \dot{H}^2}{GRH^3} \geq 0.$

Thermodynamics in Jordan Frame

- Non-equilibrium case
  - First law: $Td\hat{S} + Td_i\hat{S} = -d\hat{E} + \widehat{W}dV$
  - Second law: $\frac{d\hat{S}}{dt} + \frac{d_i\hat{S}}{dt} + \frac{d\hat{S}_t}{dt} = \frac{12\pi F\dot{H}^2}{GRH^3} \geq 0$ provided F>0.

- Equilibrium case
  - First law: $TdS = -dE + WdV$
  - Second law: $\frac{d}{dt}(S + S_t) = \frac{12\pi\dot{H}^2}{GRH^3} \geq 0.$

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Thermodynamics in Einstein Frame

- $\tilde{g}_{\alpha\beta}(x) = \Omega^2(x) g_{\alpha\beta}(x)$
- The action becomes

$$S = \int d^4x \sqrt{-\tilde{g}} \left[\frac{1}{2\kappa} \tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \omega \partial_\nu \omega - V(\omega) \right] + \sum_i S_{\mathrm{M}}^{(i)}[\omega, \tilde{g}_{\mu\nu}, \Psi_{\mathrm{M}}]$$

- The RW metric becomes

$$d\tilde{t} = \Omega\, dt$$
$$\tilde{a}(t) = \Omega\, a(t)$$

$$d\tilde{s}^2 = -d\tilde{t}^2 + \tilde{a}^2(\tilde{t})(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2)$$

➢ The EoMs

$$\tilde{G}_{\mu\nu} = \sum \kappa \left(\tilde{T}_{\mu\nu}^{(i)} + \tilde{t}_{\mu\nu}^{(i)} \right) + \kappa \tilde{\mathcal{T}}_{\mu\nu},$$

$$\frac{\partial \mathcal{L}_i}{\partial \phi} + \bar{V}_\alpha \frac{\partial \mathcal{L}_i}{\partial(\partial_\alpha \phi)} = 0,$$

$$\frac{1}{\sqrt{-\tilde{g}}} \partial_\mu \left(\sqrt{-\tilde{g}}\, \tilde{g}^{\mu\nu} \partial_\nu \omega \right) - \frac{\partial V}{\partial \omega} + \sum_i \left(\frac{1}{\sqrt{-\tilde{g}}} \frac{\delta \mathcal{L}_i}{\delta \omega} - \frac{1}{\alpha} \tilde{g}^{\mu\nu} \tilde{t}_{\mu\nu}^{(i)} \right) = 0.$$

Modified Friedmann equations

- $\widetilde{H}^2 = \frac{8\pi G}{3}\left((\tilde{\bar{\rho}}_{\mathrm{M}} + \tilde{\rho}_\omega) + \tilde{\lambda}\tilde{\bar{\rho}}_{\mathrm{M}}\right) \coloneqq \frac{8\pi G}{3}\tilde{\rho}_t$
- $\widetilde{H}' = -4\pi G\left((\tilde{\bar{\rho}}_{\mathrm{M}} + \tilde{\rho}_\omega + \tilde{P}_r + \tilde{P}_\omega) + \tilde{\lambda}\left(1 + \widetilde{\omega}^{(\mathrm{in})}\right)\tilde{\bar{\rho}}_{\mathrm{M}}\right)$
  $\coloneqq -4\pi G(\tilde{\rho}_t + \tilde{P}_t)$

with $\tilde{\bar{\rho}}_{\mathrm{M}} = \tilde{\rho}_{\mathrm{m}} + \tilde{\rho}_{\mathrm{r}}$, $\tilde{\rho}_\omega = \frac{1}{2}\omega'^2 + V(\omega)$, and $\hat{P}_\omega = \frac{1}{2}\omega'^2 - V(\omega)$.

➢It can be shown that the total energy and pressure obeys the continuity equation:

$$\tilde{\rho}_t' + 3\widetilde{H}\left(\tilde{\rho}_t + \tilde{P}_t\right) = 0.$$

➢Thermodynamics in Einstein frame can be considered as an equilibrium description.

1^{st} & 2^{nd} Law of Thermodynamics (in Einstein frame)

$\tilde{S}$: Horizon entropy in Einstein frame

$\widetilde{W}$: work density in Einstein frame

- First law: $\tilde{T}d\tilde{S} = -d\tilde{E} + \widetilde{W}d\tilde{V}$

$\tilde{T}$: Temperature in Einstein frame

$\tilde{E} = \tilde{\rho}_t\tilde{V}$(Total energy within the horizon in Einstein frame)

- Second law: $\frac{d}{d\tilde{t}}\left(\tilde{S} + \tilde{S}_t\right) = \frac{12\pi\widetilde{H}'^2}{G\tilde{R}\widetilde{H}^3} \geq 0$

$d\tilde{S}_t$:Entropy change rate of ordinary matter and induced matter within the horizon

Summary

- Consider the physical metric directly coupled to matter.
- Consider the simple case: $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$.
- Interpret the effects of f(R) deviated from GR as dark energy.

Verify the first and second laws of thermodynamics in f(R) gravity with disformal transformation in both Jordan and Einstein frames.

THANK YOU FOR YOUR ATTENTION!

Back-up slides

- Consider only simple case : $\gamma_{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{diag}(-1, +1, +1, +1)$.

  Thus, the disformal transformation becomes

$$\eta_{\alpha\beta} = A(\phi, X)g_{\alpha\beta} + B(\phi, X)\partial_\alpha\phi\partial_\beta\phi$$

- $\delta\eta_{\alpha\beta} = 0 \;\Rightarrow\; \delta\left(\partial_\beta\phi\right) = \bar{V}_\beta A_{\mu\nu}\,\delta g^{\mu\nu} + \bar{V}_\beta\,\delta\phi$

where $\bar{V}_\beta = (\partial_\beta\phi)\,\frac{4\partial_\phi A - 2X\,\partial_\phi B}{(2X)(-4A_{,X}+2B+2XB_{,X})}$,

$$A_{\mu\nu} = \frac{-Ag_{\mu\nu}-A_{,X}\partial_\mu\phi\partial_\nu\phi+XB_{,X}\partial_\mu\phi\partial_\nu\phi}{4\partial_\phi A-2X\partial_\phi B}.$$

1st & 2nd Law of Thermodynamics (Non-equilibrium case)

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- $\dot{\hat{\rho}}_{\mathrm{t}} + 3H\left(\hat{\rho}_{\mathrm{t}} + \hat{P}_{\mathrm{t}}\right) = \frac{3H^2\dot{F}}{8\pi G}$ $\Rightarrow$ Non-equilibrium Thermodynamics!

$$T = \frac{\kappa_s}{2\pi} = \frac{1}{2\pi\bar{r}_A}\left(1 - \frac{\dot{\bar{r}}_A}{2H\bar{r}_A}\right)$$

$$\hat{S} = \frac{FA}{4G}$$

$$W = \frac{1}{2}(\hat{\rho}_t - \hat{P}_t)$$

- First law: $Td\hat{S} + Td_i\hat{S} = -dE + \widehat{W}dV,$

$$d_i\hat{S} = -\frac{1}{T}\frac{\bar{r}_A}{2G}(1 + 2\pi\bar{r}_A T)dF$$

$$\hat{E} = V\hat{\rho}_t$$

- Second law: $\frac{d\hat{S}}{dt} + \frac{d_i\hat{S}}{dt} + \frac{d\hat{S}_t}{dt} = \frac{12\pi F\dot{H}^2}{GRH^3} \geq 0.$

$$d\hat{S}_t = \frac{1}{T}[d(\hat{\rho}_t V) + \hat{P}_t dV]$$

1st & 2nd Law of Thermodynamics (Equilibrium case)

- $\dot{\hat{\rho}}_{\mathrm{t}} + 3H\left(\hat{\rho}_{\mathrm{t}} + \hat{P}_{\mathrm{t}}\right) = 0$

$$W = \frac{1}{2}(\hat{\rho}_t - \hat{P}_t)$$

$$S = \frac{A}{4G}$$

$$W = \frac{1}{2}(\hat{\rho}_t - \hat{P}_t)$$

- First law: $TdS = -dE + WdV$

$$T = \frac{\kappa_s}{2\pi} = \frac{1}{2\pi\tilde{r}_A}\left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right)$$

$$\hat{E} = V\hat{\rho}_t$$

- Second law of thmodynamics: $\frac{d}{dt}(S + S_t) = \frac{12\pi\dot{H}^2}{GRH^3} \geq 0.$

$$dS_t = \frac{1}{T}[d(\rho_t V) + P_t dV]$$

1^{st} & 2^{nd} Law of Thermodynamics (in Einstein frame)

$$\tilde{S} = \frac{\tilde{A}}{4G}$$

$$\widetilde{W} = \frac{1}{2}(\tilde{\rho}_t - \tilde{P}_t)$$

- First law: $\tilde{T}d\tilde{S} = -d\tilde{E} + \widetilde{W}d\tilde{V}$

$$\tilde{T} = \frac{1}{2\pi\tilde{r}_A}\left(1 - \frac{\dot{\tilde{r}}_A}{2\,\widetilde{H}\tilde{r}_A}\right)$$

$$\tilde{E} = \tilde{\rho}_t\tilde{V}$$

- Second law: $\frac{d}{d\tilde{t}}\left(\tilde{S} + \tilde{S}_t\right) = \frac{12\pi\widetilde{H}'^2}{G\tilde{R}\widetilde{H}^3} \geq 0$

$$d\tilde{S}_t = \frac{1}{\tilde{T}}\left(d\left(\tilde{\rho}_t\tilde{V}\right) + \tilde{P}_t d\tilde{V}\right),$$